京大理系　1964前期　⑤

(イ) $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$+)\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$　　$\sin\alpha\cos\beta=\frac{1}{2}\{\sin(\alpha+\beta)+\sin(\alpha-\beta)\}$

$f_n(x)=\frac{1}{2}\{\sin(\frac{1}{2}a_{n+1}\cancel{-\frac{1}{2}a_n}+x+\frac{1}{2}a_{n+1}\cancel{+\frac{1}{2}a_n})+\sin(\cancel{\frac{1}{2}a_{n+1}}-\frac{1}{2}a_n-x\cancel{-\frac{1}{2}a_{n+1}}-\frac{1}{2}a_n)\}$

$=\frac{1}{2}\{\sin(a_{n+1}+x)-\sin(a_n+x)\}$ より

$\sum_{n=1}^{N}f_n(x)=\frac{1}{2}\{\cancel{\sin(a_2+x)}-\sin(a_1+x)\}+\frac{1}{2}\{\cancel{\sin(a_3+x)}-\cancel{\sin(a_2+x)}\}+\cdots+\frac{1}{2}\{\sin(a_{N+1}+x)-\cancel{\sin(a_N+x)}\}$

$=\frac{1}{2}\sin a_{N+1}\cos x+\frac{1}{2}\cos a_{N+1}\sin x-\frac{1}{2}\sin(a_1+x)$

よって $n\to\infty$ のとき $\sin a_n$, $\cos a_n$ が収束すればよい.

(ロ) $n\to\infty$ のとき $\sin a_n\to\alpha$, $\cos a_n\to\beta$ とする

$F(x)=\frac{1}{2}\alpha\cos x+\frac{1}{2}\beta\sin x-\frac{1}{2}\sin(a_1+x)$

$\int_0^{\frac{\pi}{2}}F(x)dx=\left[\frac{1}{2}\alpha\sin x-\frac{1}{2}\beta\cos x+\frac{1}{2}\cos(a_1+x)\right]_0^{\frac{\pi}{2}}=\frac{1}{2}\alpha+\frac{1}{2}\cos(a_1+\frac{\pi}{2})+\frac{1}{2}\beta-\frac{1}{2}\cos a_1=\frac{1}{2}\alpha+\frac{1}{2}\beta-\frac{1}{2}\sin a_1-\frac{1}{2}\cos a_1$ —①

$\int_0^{\frac{\pi}{2}}f_n(x)dx=\left[-\frac{1}{2}\cos(a_{n+1}+x)+\frac{1}{2}\cos(a_n+x)\right]_0^{\frac{\pi}{2}}=-\frac{1}{2}\cos(a_{n+1}+\frac{\pi}{2})+\frac{1}{2}\cos(a_n+\frac{\pi}{2})+\frac{1}{2}\cos a_{n+1}-\frac{1}{2}\cos a_n$

$=\frac{1}{2}\sin a_{n+1}+\frac{1}{2}\cos a_{n+1}-\frac{1}{2}\sin a_n-\frac{1}{2}\cos a_n$

$\sum_{n=1}^{N}\left(\int_0^{\frac{\pi}{2}}f_n(x)dx\right)=\cancel{\frac{1}{2}\sin a_2}+\cancel{\frac{1}{2}\cos a_2}-\frac{1}{2}\sin a_1-\frac{1}{2}\cos a_1+\cancel{\frac{1}{2}\sin a_3}+\cancel{\frac{1}{2}\cos a_3}-\cancel{\frac{1}{2}\sin a_2}-\cancel{\frac{1}{2}\cos a_2}+\cdots+\frac{1}{2}\sin a_{N+1}+\frac{1}{2}\cos a_{N+1}-\cancel{\frac{1}{2}\sin a_N}-\cancel{\frac{1}{2}\cos a_N}$

$=\frac{1}{2}\sin a_{N+1}+\frac{1}{2}\cos a_{N+1}-\frac{1}{2}\sin a_1-\frac{1}{2}\cos a_1$

$\sum_{n=1}^{\infty}\left(\int_0^{\frac{\pi}{2}}f_n(x)dx\right)=\frac{1}{2}\alpha+\frac{1}{2}\beta-\frac{1}{2}\sin a_1-\frac{1}{2}\cos a_1$ —②

①②より $\int_0^{\frac{\pi}{2}}F(x)dx=\sum_{n=1}^{\infty}\left(\int_0^{\frac{\pi}{2}}f_n(x)dx\right)$