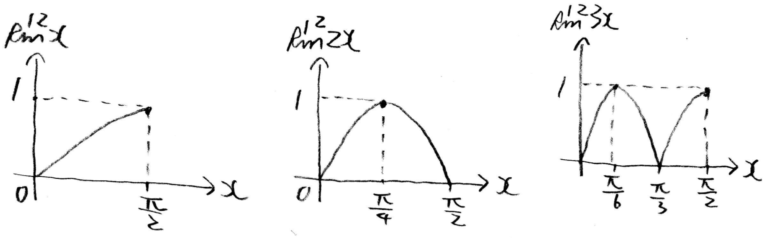
東工大　1999後期　①



$\frac{k\pi}{2n} \leqq x \leqq \frac{(k+1)\pi}{2n}$ $(k=0,1,2\cdots n-1)$ のとき.　$\frac{1}{1+\frac{(k+1)\pi}{2n}} \leqq \frac{1}{1+x} \leqq \frac{1}{1+\frac{k\pi}{2n}}$ であるから

$$\sum_{k=0}^{n-1} \frac{1}{1+\frac{(k+1)\pi}{2n}} \int_{\frac{k\pi}{2n}}^{\frac{(k+1)\pi}{2n}} \sin^2 nx\,dx \leqq \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{1+x}\,dx \leqq \sum_{k=0}^{n-1} \frac{1}{1+\frac{k\pi}{2n}} \int_{\frac{k\pi}{2n}}^{\frac{(k+1)\pi}{2n}} \sin^2 nx\,dx \quad \text{—①}$$

$$\int_{\frac{k\pi}{2n}}^{\frac{(k+1)\pi}{2n}} \sin^2 nx\,dx = \int_{\frac{k\pi}{2n}}^{\frac{(k+1)\pi}{2n}} \frac{1-\cos 2nx}{2}\,dx = \frac{1}{2}\left[x - \frac{1}{2n}\sin 2nx\right]_{\frac{k\pi}{2n}}^{\frac{(k+1)\pi}{2n}} = \frac{1}{2}\left\{\frac{(k+1)\pi}{2n} - \frac{k\pi}{2n}\right\} = \frac{\pi}{4n}$$ より. ①より

$$\frac{\pi}{4}\sum_{k=0}^{n-1} \frac{1}{1+\frac{(k+1)\pi}{2n}}\frac{1}{n} \leqq \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{1+x}\,dx \leqq \frac{\pi}{4}\sum_{k=0}^{n-1} \frac{1}{1+\frac{k\pi}{2n}}\frac{1}{n} \quad \text{—②}$$

$$\lim_{n\to\infty} \frac{\pi}{4}\sum_{k=0}^{n-1} \frac{1}{1+\frac{(k+1)\pi}{2n}}\frac{1}{n} = \lim_{n\to\infty} \frac{\pi}{4}\sum_{k=1}^{n} \frac{1}{1+\frac{\pi}{2}\frac{k}{n}}\frac{1}{n} = \lim_{n\to\infty} \frac{\pi}{4}\left(\sum_{k=0}^{n-1} \frac{1}{1+\frac{\pi}{2}\frac{k}{n}}\frac{1}{n} - \frac{1}{n} + \frac{1}{1+\frac{\pi}{2}}\frac{1}{n}\right)$$

$$= \frac{\pi}{4}\int_0^1 \frac{1}{1+\frac{\pi}{2}x}\,dx = \frac{\pi}{4}\left[\frac{2}{\pi}\log\left(1+\frac{\pi}{2}x\right)\right]_0^1 = \frac{1}{2}\log\left(1+\frac{\pi}{2}\right) \quad \text{—③}$$

$$\lim_{n\to\infty} \frac{\pi}{4}\sum_{k=0}^{n-1} \frac{1}{1+\frac{k\pi}{2n}}\frac{1}{n} = \frac{\pi}{4}\int_0^1 \frac{1}{1+\frac{\pi}{2}x}\,dx = \cdots = \frac{1}{2}\log\left(1+\frac{\pi}{2}\right) \quad \text{—④}$$

②③④より. はさみうちの原理 より.　$\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{1+x}\,dx = \frac{1}{2}\log\left(1+\frac{\pi}{2}\right)$