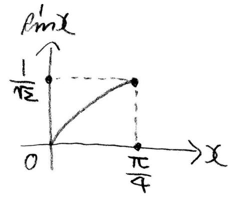
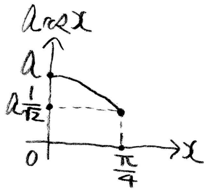
東工大 2002前期 ①





(i) $a\frac{1}{\sqrt{2}} \geqq \frac{1}{\sqrt{2}}$, $a \geqq 1$ のとき

$0 \leqq x \leqq \frac{\pi}{4}$ で $a\cos x \geqq \sin x$, $\sin x - a\cos x \leqq 0$ より

$f(a)=\int_0^{\frac{\pi}{4}}(-\sin x + a\cos x)\,dx = \left[\cos x + a\sin x\right]_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + a\frac{1}{\sqrt{2}} - 1$

よって $f(a)$ は $a=1$ のとき最小値 $\sqrt{2}-1$ をとる。

(ii) $a \leqq 0$ のとき

$0 \leqq x \leqq \frac{\pi}{4}$ で $\sin x \geqq a\cos x$, $\sin x - a\cos x \geqq 0$ より

$f(a)=\int_0^{\frac{\pi}{4}}(\sin x - a\cos x)\,dx = \left[-\cos x - a\sin x\right]_0^{\frac{\pi}{4}} = -\frac{1}{\sqrt{2}} - a\frac{1}{\sqrt{2}} + 1$

よって $f(a)$ は $a=0$ のとき最小値 $1-\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(\sqrt{2}-1)$ をとる。

(iii) $0 \leqq a \leqq 1$ のとき

α は $\sin\alpha = a\cos\alpha$, $0 \leqq \alpha \leqq \frac{\pi}{4}$ を満たすとすると

$0 \leqq x \leqq \alpha$ で $a\cos x \geqq \sin x$, $\sin x - a\cos x \leqq 0$, $\alpha \leqq x \leqq \frac{\pi}{4}$ で $\sin x \geqq a\cos x$, $\sin x - a\cos x \geqq 0$

$f(a)=\int_0^{\alpha}(-\sin x + a\cos x)\,dx + \int_{\alpha}^{\frac{\pi}{4}}(\sin x - a\cos x)\,dx = \left[\cos x + a\sin x\right]_0^{\alpha} + \left[-\cos x - a\sin x\right]_{\alpha}^{\frac{\pi}{4}}$

$= \cos\alpha + a\sin\alpha - 1 - \frac{1}{\sqrt{2}} - a\frac{1}{\sqrt{2}} + \cos\alpha + a\sin\alpha = 2\cos\alpha + 2\frac{\sin\alpha}{\cos\alpha}\sin\alpha - \frac{\sin\alpha}{\cos\alpha}\frac{1}{\sqrt{2}} - 1 - \frac{1}{\sqrt{2}}$

$= \frac{2\sqrt{2}\cos^2\alpha + 2\sqrt{2}(1-\cos^2\alpha) - \sin\alpha}{\sqrt{2}\cos\alpha} - 1 - \frac{1}{\sqrt{2}} = \frac{2 - \frac{1}{\sqrt{2}}\sin\alpha}{\cos\alpha} - 1 - \frac{1}{\sqrt{2}}$

$g(\alpha) = \frac{2 - \frac{1}{\sqrt{2}}\sin\alpha}{\cos\alpha} - 1 - \frac{1}{\sqrt{2}}$ $\left(0 \leqq \alpha \leqq \frac{\pi}{4}\right)$ とすると

$g'(\alpha) = \frac{-\frac{1}{\sqrt{2}}\cos\alpha\cos\alpha + \left(2 - \frac{1}{\sqrt{2}}\sin\alpha\right)\sin\alpha}{\cos^2\alpha} = \frac{2\sin\alpha - \frac{1}{\sqrt{2}}}{\cos^2\alpha}$, $g'(\alpha)=0$ のとき $\sin\alpha = \frac{1}{2\sqrt{2}}$

α	0	$\cdots$	β	$\cdots$	$\frac{\pi}{4}$
$g'(\alpha)$		$-$	0	$+$	
$g(\alpha)$	$1-\frac{1}{\sqrt{2}}$	$\searrow$	$\frac{\sqrt{14}-\sqrt{2}-2}{2}$	$\nearrow$	$\sqrt{2}-1$

β は $\sin\beta = \frac{1}{2\sqrt{2}}$, $0 \leqq \beta \leqq \frac{\pi}{4}$ を満たすとすると

$g(\alpha)$ の増減表は左表のようになる。

よって $f(a)$ は最小値 $\frac{\sqrt{14}-\sqrt{2}-2}{2}$ をとる。

※ $\frac{1}{8} + \cos^2\beta = 1$, $\cos\beta = \frac{\sqrt{7}}{2\sqrt{2}}$ より

$g(\beta) = \frac{2-\frac{1}{4}}{\frac{\sqrt{7}}{2\sqrt{2}}} - 1 - \frac{1}{\sqrt{2}} = \frac{7}{4}\cdot\frac{2\sqrt{2}}{\sqrt{7}} - 1 - \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{14}}{2} - 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{14}-\sqrt{2}-2}{2}$

$g\left(\frac{\pi}{4}\right) = \frac{2-\frac{1}{2}}{\frac{1}{\sqrt{2}}} - 1 - \frac{1}{\sqrt{2}} = \frac{3}{2}\sqrt{2} - 1 - \frac{1}{\sqrt{2}} = \sqrt{2}-1$

(i)(ii)(iii) より $f(a)$ の最小値は $\frac{\sqrt{14}-\sqrt{2}-2}{2}$