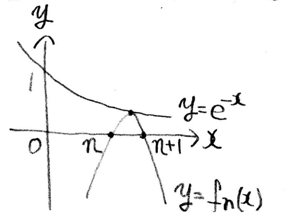
東工大 2007前期 ④

(1)

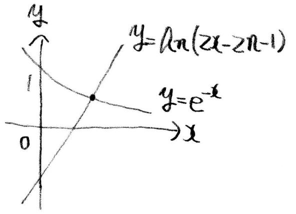


$g(x)=e^{-x}-a_n(x-n)(n+1-x)$ とする.

$g'(x)=-e^{-x}-a_n(n+1-x-x+n)=-e^{-x}+a_n(2x-2n-1)$

$g'(x)=0$ のとき $e^{-x}=a_n(2x-2n-1)$

左下図より. これを満たすxはただ1つだけ存在するから. これをx_nとする



x	$\cdots$	x_n	$\cdots$
$g'(x)$	$-$	0	$+$
$g(x)$	↘	極小	↗

$g(x)$の増減表は左表

2つの曲線が接するには. $g(x_n)=0$ であればよい.

$e^{-x_n}-a_n(x_n-n)(n+1-x_n)=0$. $a_n(2x_n-2n-1-nx_n-x_n+x_n^2+n^2+n-nx_n)=0$

$a_n\neq 0$ より. $x_n^2+(-2n+1)x_n+n^2-n-1=0$.

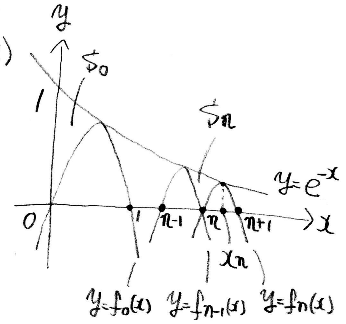
$x_n=\dfrac{2n-1\pm\sqrt{4n^2-4n+1-4n^2+4n+4}}{2}=n-\dfrac12\pm\dfrac{\sqrt5}{2}$

(i) $x_n=n-\frac12+\frac{\sqrt5}{2}$ のとき. $e^{-n+\frac12-\frac{\sqrt5}{2}}=a_n(2n-1+\sqrt5-2n-1)$. $a_n=\dfrac{e^{-n+\frac12-\frac{\sqrt5}{2}}}{-2+\sqrt5}$

(ii) $x_n=n-\frac12-\frac{\sqrt5}{2}$ のとき $e^{-n+\frac12+\frac{\sqrt5}{2}}=a_n(2n-1-\sqrt5-2n-1)$ $a_n=\dfrac{e^{-n+\frac12+\frac{\sqrt5}{2}}}{-2-\sqrt5}$ $a_n<0$ より不適

(i)(ii)より. $a_n=\dfrac{e^{-n+\frac12-\frac{\sqrt5}{2}}}{-2+\sqrt5}$

(2)



$\int_0^{x_n}e^{-x}dx=\left[-e^{-x}\right]_0^{x_n}=-e^{-x_n}+1=-e^{-n+\frac12+\frac{\sqrt5}{2}}+1$ —①

$y=f_n(x)$とx軸で囲まれる図形の面積をT_nとすると

$T_n=\int_n^{n+1}a_n(x-n)(n+1-x)dx=a_n\left[-\frac{x^3}{3}+(2n+1)\frac{x^2}{2}-n(n+1)x\right]_n^{n+1}$

$=a_n\left\{-\frac13(n+1)^3+\frac12(2n+1)(n+1)^2-n(n+1)^2+\frac13n^3-\frac12(2n+1)n^2+n^2(n+1)\right\}$

$=a_n\left\{-\frac13(3n^2+3n+1)+\frac12(2n+1)(2n+1)+n(n+1)(-1)\right\}$

$=a_n\left(-n^2-n-\frac13+2n^2+2n+\frac12-n^2-n\right)=\frac16a_n$

$T_0+T_1+\cdots T_{n-1}=\frac16(a_0+a_1+\cdots+a_{n-1})=\dfrac{e^{\frac12-\frac{\sqrt5}{2}}}{6(-2+\sqrt5)}\{1+e^{-1}+\cdots+e^{-(n-1)}\}=\dfrac{e^{\frac12-\frac{\sqrt5}{2}}}{6(-2+\sqrt5)}\dfrac{1-e^{-n}}{1-e^{-1}}$ —②

$\int_n^{x_n}a_n(x-n)(n+1-x)dx=a_n\left[-\frac{x^3}{3}+(2n+1)\frac{x^2}{2}-n(n+1)x\right]_n^{n-\frac12+\frac{\sqrt5}{2}}$

$=a_n\left\{-\frac13\left(n+\frac{\sqrt5-1}{2}\right)^3+\frac12(2n+1)\left(n+\frac{\sqrt5-1}{2}\right)^2-n(n+1)\left(n+\frac{\sqrt5-1}{2}\right)+\frac13n^3-\frac12(2n+1)n^2+n^2(n+1)\right\}$ ※ $\alpha=\frac{\sqrt5-1}{2}$

$=a_n\left\{-\frac13(3n^2\alpha+3n\alpha^2+\alpha^3)+\frac12(2n+1)(2n\alpha+\alpha^2)+n(n+1)(-\alpha)\right\}=a_n\left(-n^2\alpha-n\alpha^2-\frac13\alpha^3+2n^2\alpha+n\alpha^2+n\alpha+\frac12\alpha^2-n^2\alpha-n\alpha\right)$

$=\dfrac{e^{-n+\frac12-\frac{\sqrt5}{2}}}{-2+\sqrt5}\left(-\frac13\alpha^3+\frac12\alpha^2\right)$ —③

①②③より. $S_0+S_1+\cdots+S_n=e^{-n+\frac12+\frac{\sqrt5}{2}}+1-\dfrac{e^{\frac12-\frac{\sqrt5}{2}}}{6(-2+\sqrt5)}\dfrac{1-\frac{1}{e^n}}{1-\frac1e}-\dfrac{e^{-n+\frac12-\frac{\sqrt5}{2}}}{-2+\sqrt5}\left(-\frac13\alpha^3+\frac12\alpha^2\right)$

$\lim_{n\to\infty}(S_0+S_1+\cdots+S_n)=1-\dfrac{e^{\frac12-\frac{\sqrt5}{2}}}{6(-2+\sqrt5)}\dfrac{1}{1-\frac1e}=1-\dfrac{e^{\frac32-\frac{\sqrt5}{2}}(2+\sqrt5)}{6(-2+\sqrt5)(2+\sqrt5)(e-1)}=1-\dfrac{2+\sqrt5}{6(e-1)}e^{\frac{3-\sqrt5}{2}}$