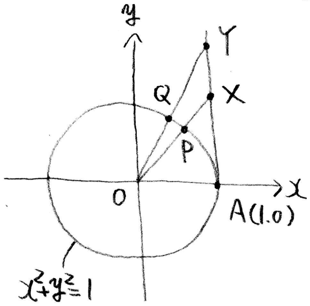
東工大　2011前期　③



Xの座標を$(1.\alpha)$ $(\alpha>0)$とするとYの座標は$(1, k\alpha)$

直線OX. OYの方程式は $y=\alpha x$, $y=k\alpha x$

$x^2+\alpha^2x^2=1$. $x=\dfrac{1}{\sqrt{1+\alpha^2}}$, $\dfrac{1}{1+\alpha^2}+y^2=1$　$y^2=\dfrac{\alpha^2}{1+\alpha^2}$, $y=\dfrac{\alpha}{\sqrt{1+\alpha^2}}$ より

Pの座標は $\left(\dfrac{1}{\sqrt{1+\alpha^2}}, \dfrac{\alpha}{\sqrt{1+\alpha^2}}\right)$

$x^2+k^2\alpha^2x^2=1$　$x=\dfrac{1}{\sqrt{1+k^2\alpha^2}}$　$\dfrac{1}{1+k^2\alpha^2}+y^2=1$　$y^2=\dfrac{k^2\alpha^2}{1+k^2\alpha^2}$　$y=\dfrac{k\alpha}{\sqrt{1+k^2\alpha^2}}$ より

Qの座標は $\left(\dfrac{1}{\sqrt{1+k^2\alpha^2}}, \dfrac{k\alpha}{\sqrt{1+k^2\alpha^2}}\right)$

△OPQの面積を$f(\alpha)$とすると $f(\alpha)=\dfrac{1}{2}\left|\dfrac{1}{\sqrt{1+\alpha^2}}\dfrac{k\alpha}{\sqrt{1+k^2\alpha^2}}-\dfrac{\alpha}{\sqrt{1+\alpha^2}}\dfrac{1}{\sqrt{1+k^2\alpha^2}}\right|=\dfrac{1}{2}\dfrac{(k-1)\alpha}{\sqrt{1+\alpha^2}\sqrt{1+k^2\alpha^2}}$

$\left\{\dfrac{2f(\alpha)}{k-1}\right\}^2=g(\alpha)$とする. $g(\alpha)=\dfrac{\alpha^2}{(1+\alpha^2)(1+k^2\alpha^2)}=\dfrac{\alpha^2}{k^2\alpha^4+(k^2+1)\alpha^2+1}=\dfrac{1}{k^2\alpha^2+\frac{1}{\alpha^2}+k^2+1}$

相加平均≧相乗平均 より $k^2\alpha^2+\dfrac{1}{\alpha^2}\geqq 2\sqrt{k^2\alpha^2\dfrac{1}{\alpha^2}}=2k$　等号は $k^2\alpha^2=\dfrac{1}{\alpha^2}$, $\alpha=\dfrac{1}{\sqrt{k}}$ のとき成立つ

よって$g(\alpha)$の最大値は $\dfrac{1}{2k+k^2+1}=\dfrac{1}{(k+1)^2}$

このとき $\left\{\dfrac{2f(\alpha)}{k-1}\right\}^2=\dfrac{1}{(k+1)^2}$, $\dfrac{2f(\alpha)}{k-1}=\dfrac{1}{k+1}$,　$f(a)=\dfrac{k-1}{2(k+1)}$

ゆえに △OPQの面積の最大値は $\dfrac{k-1}{2(k+1)}$