東工大　2012前期　④

(1) $a_2 = -\frac{1}{1+n+1} + \frac{n}{1}\sum_{i=1}^{1} a_i = -\frac{1}{n+2} + n\frac{1}{\cancel{n}(n+1)} = \frac{1}{n+1} - \frac{1}{n+2} = \frac{n+2-n-1}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}$

$a_3 = -\frac{1}{2+n+1} + \frac{n}{2}\sum_{i=1}^{2} a_i = -\frac{1}{n+3} + \frac{n}{2}\left(\frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right) = -\frac{1}{n+3} + \frac{\cancel{n}}{2}\,\frac{n+2-n}{\cancel{n}(n+2)} = \frac{1}{n+2} - \frac{1}{n+3} = \frac{n+3-n-2}{(n+2)(n+3)} = \frac{1}{(n+2)(n+3)}$

※ $\frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)}$

(2) $a_k = \frac{1}{(n+k-1)(n+k)}$ —①　　$k=1$のとき①は成り立つ —②

lを自然数とすると、$k=1, 2 \cdots l$ のとき①が成り立つと仮定すると

$a_{l+1} = -\frac{1}{l+n+1} + \frac{n}{l}\sum_{i=1}^{l} a_i = -\frac{1}{n+l+1} + \frac{n}{l}\left(\frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} - \cancel{\frac{1}{n+2}} + \cdots + \cancel{\frac{1}{n+l-1}} - \frac{1}{n+l}\right) = -\frac{1}{n+l+1} + \frac{n}{l}\,\frac{n+l-n}{n(n+l)}$

※ $\frac{1}{n+k-1} - \frac{1}{n+k} = \frac{n+k-n-k+1}{(n+k-1)(n+k)}$

$= \frac{1}{n+l} - \frac{1}{n+l+1} = \frac{n+l+1-n-l}{(n+l)(n+l+1)} = \frac{1}{\{n+(l+1)-1\}\{n+(l+1)\}}$ より、$k=l+1$ のときも①は成り立つ —③

②③より数学的帰納法より①は成り立つ。すなわち $a_k = \frac{1}{(n+k-1)(n+k)}$

(3) $b_n = \sum_{k=1}^{n}\sqrt{\frac{1}{(n+k-1)(n+k)}} = \sum_{k=1}^{n}\sqrt{\frac{1}{(1+\frac{k-1}{n})(1+\frac{k}{n})}}\,\frac{1}{n}$

$\sum_{k=1}^{n}\sqrt{\frac{1}{(1+\frac{k}{n})(1+\frac{k}{n})}}\,\frac{1}{n} \leqq b_n \leqq \sum_{k=1}^{n}\sqrt{\frac{1}{(1+\frac{k-1}{n})(1+\frac{k-1}{n})}}\,\frac{1}{n}$,　$\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}\,\frac{1}{n} \leqq b_n \leqq \sum_{k=1}^{n}\frac{1}{1+\frac{k-1}{n}}\,\frac{1}{n}$ —④

$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}\,\frac{1}{n} = \lim_{n\to\infty}\left(\sum_{k=0}^{n-1}\frac{1}{1+\frac{k}{n}}\,\frac{1}{n} + \frac{1}{2n} - \frac{1}{n}\right) = \int_0^1 \frac{1}{1+x}dx = \left[\log(1+x)\right]_0^1 = \log 2$ —⑤

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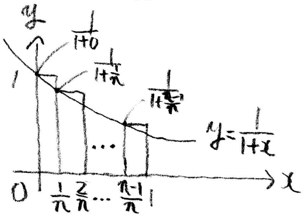
左図より



$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+\frac{k-1}{n}}\,\frac{1}{n} = \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{1+\frac{k}{n}}\,\frac{1}{n} = \int_0^1\frac{1}{1+x}dx = \left[\log(1+x)\right]_0^1 = \log 2$ —⑥

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左図より

④⑤⑥より、はさみうちの原理より $\lim_{n\to\infty} b_n = \log 2$