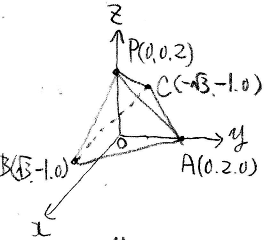
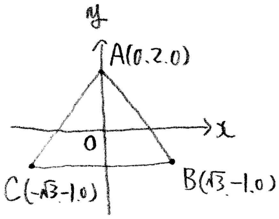
東工大　2012前期⑥

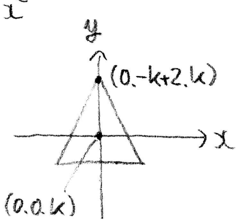




△ABCはxy平面上にあり重心が$(0.0.0)$、1辺の長さが$2\sqrt{3}$の正三角形

線分APの方程式は $(0,2,0)+t(0,-2,2)=(0,-2t+2,2t)$ $(0\leqq t\leqq 1)$

これと平面 $z=k$ $(0\leqq k\leqq 2)$ の交点は $2t=k$, $t=\frac{k}{2}$ より $(0,-k+2,k)$

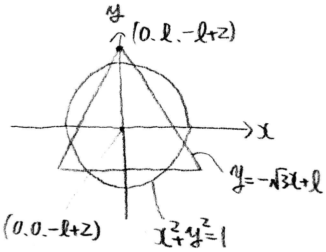


PABCを$z=k$で切った切り口は、重心が$(0.0.k)$、頂点の1つが$(0,-k+2,k)$の正三角形

これの $x^2+y^2\geqq 1$ の部分の面積を$S(k)$とする

求める体積をVとすると、$V=\int_0^2 S(k)dk$

$-k+2=l$ とおく. $\begin{array}{c|c} k & 0\to 2 \\ l & 2\to 0 \end{array}$　$\frac{dl}{dk}=-1$　$V=\int_2^0 S(l)(-1)dl=\int_0^2 S(l)dl$



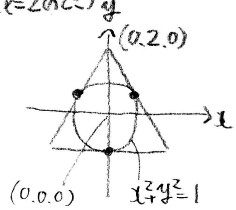
$0\leqq l<1$ のとき $S(l)=0$ より $V=\int_1^2 S(l)dl$

$1\leqq l\leqq 2$のとき $x^2+y^2=1$ と $y=-\sqrt{3}x+l$ の交点のうち、x座標が小さいほうは

$x^2+3x^2-2\sqrt{3}lx+l^2=1$, $4x^2-2\sqrt{3}lx+l^2-1=0$　$x=\frac{\sqrt{3}l\pm\sqrt{3l^2-4(l^2-1)}}{4}=\frac{\sqrt{3}l\pm\sqrt{4-l^2}}{4}$

$x=\frac{\sqrt{3}l-\sqrt{4-l^2}}{4}$, $y=\frac{-3l+\sqrt{3}\sqrt{4-l^2}+4l}{4}=\frac{l+\sqrt{3}\sqrt{4-l^2}}{4}$ より $\left(\frac{\sqrt{3}l-\sqrt{4-l^2}}{4}, \frac{l+\sqrt{3}\sqrt{4-l^2}}{4}\right)$ —①

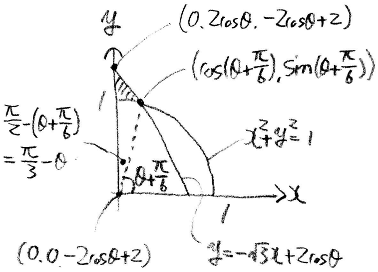
(l=2のとき)



$l=2\cos\theta$ とおく. $\begin{array}{c|c} l & 1\to 2 \\ \theta & \frac{\pi}{3}\to 0 \end{array}$,　$\frac{dl}{d\theta}=-2\sin\theta$　$V=\int_{\frac{\pi}{3}}^0 S(\theta)(-2)\sin\theta d\theta=2\int_0^{\frac{\pi}{3}} S(\theta)\sin\theta d\theta$

①は $\left(\frac{2\sqrt{3}\cos\theta-\sqrt{4-4\cos^2\theta}}{4}, \frac{2\cos\theta+\sqrt{3}\sqrt{4-4\cos^2\theta}}{4}\right)=\left(\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta, \frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)$

$=\left(\cos\theta\cos\frac{\pi}{6}-\sin\theta\sin\frac{\pi}{6}, \sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}\right)=\left(\cos(\theta+\frac{\pi}{6}), \sin(\theta+\frac{\pi}{6})\right)$ となる.



左図の斜線部の面積は 対称性より $\frac{1}{6}S(\theta)$であり

$\frac{1}{6}S(\theta)=2\cos\theta\cos(\theta+\frac{\pi}{6})\frac{1}{2}-\pi\frac{\frac{\pi}{3}-\theta}{2\pi}$

$=\frac{1}{2}\cos(2\theta+\frac{\pi}{6})+\frac{1}{2}\cos(-\frac{\pi}{6})-\frac{\pi}{6}+\frac{\theta}{2}$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$+)\ \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta$

$V=\int_0^{\frac{\pi}{3}}\left\{6\cos(2\theta+\frac{\pi}{6})+6\frac{\sqrt{3}}{2}-2\pi+6\theta\right\}\sin\theta d\theta$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$-)\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta$

$=3\int_0^{\frac{\pi}{3}}\left\{\sin(3\theta+\frac{\pi}{6})-\sin(\theta+\frac{\pi}{6})\right\}d\theta+(3\sqrt{3}-2\pi)\int_0^{\frac{\pi}{3}}\sin\theta d\theta+6\int_0^{\frac{\pi}{3}}\theta(-\cos\theta)'d\theta$

$=3\left[-\frac{1}{3}\cos(3\theta+\frac{\pi}{6})+\cos(\theta+\frac{\pi}{6})\right]_0^{\frac{\pi}{3}}+(3\sqrt{3}-2\pi)\left[-\cos\theta\right]_0^{\frac{\pi}{3}}-6\left[\theta\cos\theta\right]_0^{\frac{\pi}{3}}+6\int_0^{\frac{\pi}{3}}\cos\theta d\theta$

$=-\cos\frac{7}{6}\pi+\frac{\sqrt{3}}{2}-3\frac{\sqrt{3}}{2}+(3\sqrt{3}-2\pi)(-\frac{1}{2}+1)-6\frac{\pi}{3}\frac{1}{2}+6\left[\sin\theta\right]_0^{\frac{\pi}{3}}$

$=\frac{\sqrt{3}}{2}-\sqrt{3}+\frac{3}{2}\sqrt{3}-\pi-\pi+6\frac{\sqrt{3}}{2}=\frac{1-2+3+6}{2}\sqrt{3}-2\pi=4\sqrt{3}-2\pi$