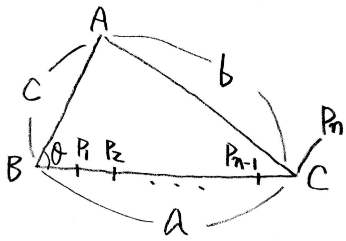
東大理科　1978前期　5



$\angle ABC=\theta$ とする.

$$AP_k^2=\left(k\frac{a}{n}\right)^2+c^2-2k\frac{a}{n}c\cos\theta=\frac{a^2}{n^2}k^2-\frac{2ac\cos\theta}{n}k+c^2$$

$$AP_1^2+AP_2^2+\cdots+AP_n^2=\frac{a^2}{n^2}\frac{1}{6}n(n+1)(2n+1)-\frac{2ac\cos\theta}{n}\frac{1}{2}n(n+1)+c^2n$$

$$=\frac{a^2}{6}\left(1+\frac{1}{n}\right)(2n+1)-ac\cos\theta(n+1)+c^2n$$

$$\lim_{n\to\infty}\frac{1}{n}(AP_1^2+AP_2^2+\cdots+AP_n^2)=\lim_{n\to\infty}\left\{\frac{a^2}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-ac\cos\theta\left(1+\frac{1}{n}\right)+c^2\right\}$$

$$=\frac{a^2}{3}-ac\cos\theta+c^2=\frac{1}{3}a^2-\frac{1}{2}a^2+\frac{1}{2}b^2-\frac{1}{2}c^2+c^2=-\frac{1}{6}a^2+\frac{1}{2}b^2+\frac{1}{2}c^2$$

※ $b^2=a^2+c^2-2ac\cos\theta$.　$ac\cos\theta=\frac{1}{2}a^2-\frac{1}{2}b^2+\frac{1}{2}c^2$