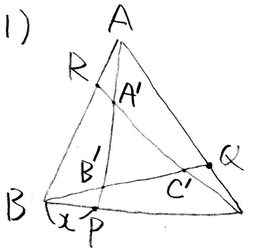
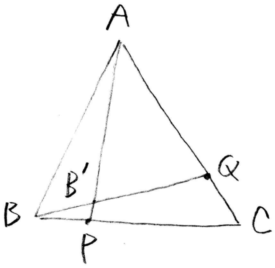
東大理科　1980前期　①

(1)




$AB':B'P=\alpha:1-\alpha$, $BB':B'Q=\beta:1-\beta$ とする.

$\overrightarrow{CB'}=\overrightarrow{CA}+\alpha\overrightarrow{AP}=\overrightarrow{CA}+\alpha\{(1-x)\overrightarrow{CB}-\overrightarrow{CA}\}$

$=(-\alpha+1)\overrightarrow{CA}+\alpha(-x+1)\overrightarrow{CB}$ —①

$\overrightarrow{CB'}=\overrightarrow{CB}+\beta\overrightarrow{BQ}=\overrightarrow{CB}+\beta(x\overrightarrow{CA}-\overrightarrow{CB})$

$=\beta x\overrightarrow{CA}+(-\beta+1)\overrightarrow{CB}$ —②

$\overrightarrow{CA}$, $\overrightarrow{CB}$ は1次独立であるから ①②より $\begin{cases}-\alpha+1=x\beta\\(-x+1)\alpha=-\beta+1\end{cases}$

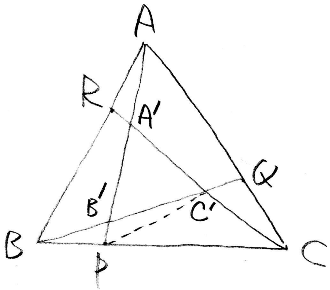
$(-x+1)\alpha=-\dfrac{-\alpha+1}{x}+1$, $(-x^2+x)\alpha=\alpha-1+x$, $\alpha=\dfrac{-x+1}{x^2-x+1}$, $\beta=\dfrac{x\cancel{-1}+x^2\cancel{-x+1}}{x^2-x+1}\dfrac{1}{\cancel{x}}=\dfrac{x}{x^2-x+1}$

余弦定理より $AP^2=1+x^2-2x\dfrac{1}{\cancel{2}}=x^2-x+1$, $AP=\sqrt{x^2-x+1}$, 対称性より $BQ=\sqrt{x^2-x+1}$


よって $BB'=\beta BQ=\dfrac{x}{\sqrt{x^2-x+1}}$, $PB'=(1-\alpha)AP=\dfrac{x^2-x\cancel{+1}+x\cancel{-1}}{x^2-x+1}\sqrt{x^2-x+1}=\dfrac{x^2}{\sqrt{x^2-x+1}}$

(2) 対称性より, $AA'=BB'=CC'=\dfrac{x}{\sqrt{x^2-x+1}}$, $B'P=C'Q=A'R=\dfrac{x^2}{\sqrt{x^2-x+1}}$

$A'B'=B'C'=C'A'=\sqrt{x^2-x+1}-\dfrac{x}{\sqrt{x^2-x+1}}-\dfrac{x^2}{\sqrt{x^2-x+1}}=\dfrac{\cancel{x^2}-x+1-x\cancel{-x^2}}{\sqrt{x^2-x+1}}=\dfrac{-2x+1}{\sqrt{x^2-x+1}}$




$\dfrac{\triangle APC}{\triangle ABC}=1-x$　$\dfrac{\triangle A'PC}{\triangle APC}=\dfrac{\dfrac{-2x+1+x^2}{\sqrt{x^2-x+1}}}{\sqrt{x^2-x+1}}=\dfrac{x^2-2x+1}{x^2-x+1}$

$\dfrac{\triangle A'PC'}{\triangle A'PC}=\dfrac{-2x+1}{x-2x+1}=\dfrac{-2x+1}{-x+1}$　$\dfrac{\triangle A'B'C'}{\triangle A'PC'}=\dfrac{-2x+1}{-2x+1+x^2}=\dfrac{-2x+1}{x^2-2x+1}$

$\dfrac{\triangle A'B'C'}{\triangle ABC}=\dfrac{\triangle APC}{\triangle ABC}\dfrac{\triangle A'PC}{\triangle APC}\dfrac{\triangle A'PC'}{\triangle A'PC}\dfrac{\triangle A'B'C'}{\triangle A'PC'}=\cancel{(-x+1)}\dfrac{\cancel{x^2-2x+1}}{x^2-x+1}\dfrac{-2x+1}{\cancel{-x+1}}\dfrac{-2x+1}{\cancel{x^2-2x+1}}=\dfrac{4x^2-4x+1}{x^2-x+1}$

$\dfrac{\triangle A'B'C'}{\triangle ABC}=\dfrac{1}{2}$ のとき $\dfrac{4x^2-4x+1}{x^2-x+1}=\dfrac{1}{2}$, $8x^2-8x+2=x^2-x+1$, $7x^2-7x+1=0$

$x=\dfrac{7\pm\sqrt{49-28}}{14}=\dfrac{7\pm\sqrt{21}}{14}$　$x<\dfrac{1}{2}$ より. $x=\dfrac{7-\sqrt{21}}{14}$