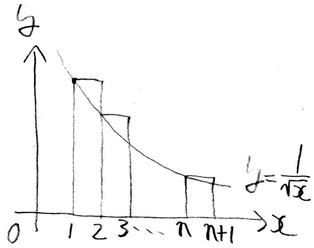
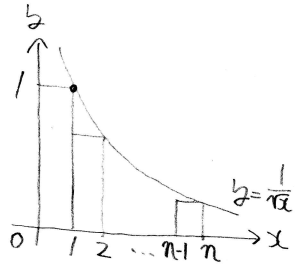
東大理科　1990前期　①



上図より $a_n > \int_1^{n+1} \frac{1}{\sqrt{x}}dx$

$= \left[2\sqrt{x}\right]_1^{n+1} = 2\sqrt{n+1}-2$ —①



上図より $a_n < 1+\int_1^{n} \frac{1}{\sqrt{x}}dx$

$=1+\left[2\sqrt{x}\right]_1^{n} = 1+2\sqrt{n}-2 = 2\sqrt{n}-1$ —②

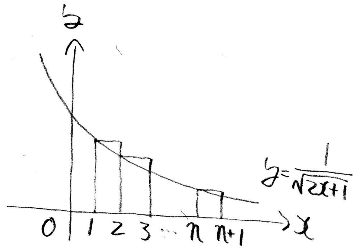
$(\sqrt{x})' = \frac{1}{2}\frac{1}{\sqrt{x}}$ より $(2\sqrt{x})' = \frac{1}{\sqrt{x}}$

①より

$\lim_{n\to\infty}(2\sqrt{n+1}-2)=\infty$

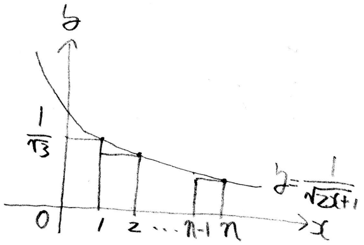
であるから

$\lim_{n\to\infty} a_n = \infty$



上図より $b_n > \int_1^{n+1} \frac{1}{\sqrt{2x+1}}dx$

$= \left[\sqrt{2x+1}\right]_1^{n+1} = \sqrt{2n+3}-\sqrt{3}$ —③



上図より $b_n < \frac{1}{\sqrt{3}} + \int_1^{n} \frac{1}{\sqrt{2x+1}}dx$

$= \frac{1}{\sqrt{3}} + \left[\sqrt{2x+1}\right]_1^{n} = \frac{1}{\sqrt{3}} + \sqrt{2n+1}-\sqrt{3}$ —④

$(\sqrt{2x+1})' = \frac{1}{2}\frac{2}{\sqrt{2x+1}}$

①②③④より

$$\frac{\sqrt{2n+3}-\sqrt{3}}{2\sqrt{n}-1} < \frac{b_n}{a_n} < \frac{\frac{1}{\sqrt{3}}+\sqrt{2n+1}-\sqrt{3}}{2\sqrt{n+1}-2}$$

$$\lim_{n\to\infty}\frac{\sqrt{2n+3}-\sqrt{3}}{2\sqrt{n}-1} = \lim_{n\to\infty}\frac{\sqrt{2+\frac{3}{n}}-\frac{\sqrt{3}}{\sqrt{n}}}{2-\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{2}},\quad \lim_{n\to\infty}\frac{\frac{1}{\sqrt{3}}+\sqrt{2n+1}-\sqrt{3}}{2\sqrt{n+1}-2} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{3n}}+\sqrt{2+\frac{1}{n}}-\frac{\sqrt{3}}{\sqrt{n}}}{2\sqrt{1+\frac{1}{n}}-\frac{2}{\sqrt{n}}} = \frac{1}{\sqrt{2}}$$ であるから

はさみうちの原理より $\lim_{n\to\infty}\frac{b_n}{a_n} = \frac{1}{\sqrt{2}}$