東大理科 1990前期 ④

(1) $M=\dfrac{1}{1+a^2}\begin{pmatrix}1&-a\\a&1\end{pmatrix}$, $\begin{pmatrix}x_1\\y_1\end{pmatrix}=\dfrac{1}{1+a^2}\begin{pmatrix}1&-a\\a&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\dfrac{1}{1+a^2}\begin{pmatrix}1\\a\end{pmatrix}$, $\begin{pmatrix}x_2\\y_2\end{pmatrix}=\dfrac{1}{(1+a^2)^2}\begin{pmatrix}1&-a\\a&1\end{pmatrix}\begin{pmatrix}1\\a\end{pmatrix}=\dfrac{1}{(1+a^2)^2}\begin{pmatrix}1-a^2\\2a\end{pmatrix}$

$\overrightarrow{P_0P_1}=\left(\dfrac{1}{1+a^2}-1,\ \dfrac{a}{1+a^2}\right)$, $\overrightarrow{P_0P_2}=\left\{\dfrac{1-a^2}{(1+a^2)^2}-1,\ \dfrac{2a}{(1+a^2)^2}\right\}$

$\triangle P_0P_1P_2$の面積は $\dfrac{1}{2}\left|\left(\dfrac{1}{1+a^2}-1\right)\dfrac{2a}{(1+a^2)^2}-\dfrac{a}{1+a^2}\left\{\dfrac{1-a^2}{(1+a^2)^2}-1\right\}\right|=\dfrac{1}{2}\left|\dfrac{2a}{(1+a^2)^3}-\dfrac{2a}{(1+a^2)^2}-\dfrac{a-a^3}{(1+a^2)^3}+\dfrac{a}{1+a^2}\right|$

$=\dfrac{1}{2}\left|\dfrac{2a-2a(1+a^2)-a+a^3+a(1+a^2)^2}{(1+a^2)^3}\right|=\dfrac{1}{2}\left|\dfrac{\cancel{2a}-\cancel{2a}-\cancel{2a^3}-\cancel{a}+a^3+\cancel{a}+\cancel{2a^3}+a^5}{(1+a^2)^3}\right|=\dfrac{1}{2}\dfrac{a^5+a^3}{(a^2+1)^3}$

$f(a)=\dfrac{a^5+a^3}{(a^2+1)^3}$, $a>0$ が最大値を取るaの値を求めればよい.

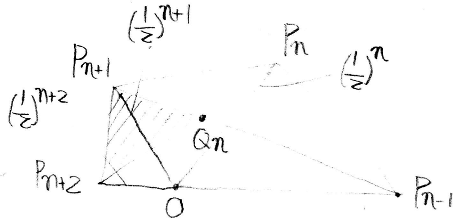
$f'(a)=\dfrac{(5a^4+3a^2)(a^2+1)^3-(a^5+a^3)3(a^2+1)^2\cdot 2a}{(a^2+1)^6}=\dfrac{a^2(a^2+1)^2\{(5a^2+3)(a^2+1)-(a^3+a)6a\}}{(a^2+1)^6}=\dfrac{a^2(5a^4+8a^2+3-6a^4-6a^2)}{(a^2+1)^4}$

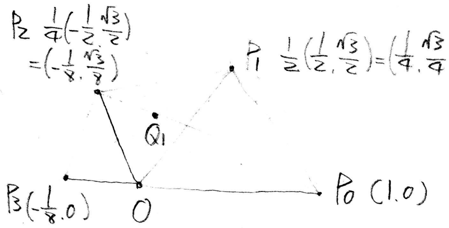
$=\dfrac{-a^2(a^4-2a^2-3)}{(a^2+1)^4}=\dfrac{-a^2(a^2-3)(a^2+1)}{(a^2+1)^4}$

a	$\cdots$	$\sqrt{3}$	$\cdots$
$f'(a)$	$+$	0	$-$
$f(a)$	$\nearrow$	最大	$\searrow$

$f(a)$の増減表は左図のようになるから
$a=\sqrt{3}$

(2) $M=\begin{pmatrix}\frac{1}{4}&-\frac{\sqrt{3}}{4}\\\frac{\sqrt{3}}{4}&\frac{1}{4}\end{pmatrix}=\dfrac{1}{2}\begin{pmatrix}\cos\frac{\pi}{3}&-\sin\frac{\pi}{3}\\\sin\frac{\pi}{3}&\cos\frac{\pi}{3}\end{pmatrix}$, Mは$\dfrac{\pi}{3}$回転して$\dfrac{1}{2}$倍することを表す





左上図のようにQ_nを取る. $\dfrac{P_{n+2}Q_n}{P_{n+2}P_n}=\dfrac{P_3Q_1}{P_3P_1}$

直線P_3P_1の方程式は $y=\dfrac{\frac{\sqrt{3}}{4}}{\frac{3}{8}}\left(x+\dfrac{1}{8}\right)=\dfrac{2\sqrt{3}}{3}x+\dfrac{\sqrt{3}}{12}$, 直線$P_2P_0$の方程式は $y=-\dfrac{\frac{\sqrt{3}}{8}}{\frac{9}{8}}(x-1)=-\dfrac{\sqrt{3}}{9}x+\dfrac{\sqrt{3}}{9}$

$\dfrac{2\sqrt{3}}{3}x+\dfrac{\sqrt{3}}{12}=-\dfrac{\sqrt{3}}{9}x+\dfrac{\sqrt{3}}{9}$, $\dfrac{7\sqrt{3}}{9}x=\dfrac{\sqrt{3}}{36}$, $x=\dfrac{1}{28}$ よりQ_1のx座標は$\dfrac{1}{28}$, $\dfrac{P_{n+2}Q_n}{P_{n+2}P_n}=\dfrac{P_3Q_1}{P_3P_1}=\dfrac{\frac{1}{8}+\frac{1}{28}}{\frac{3}{8}}=\dfrac{8}{3}\cdot\dfrac{7+2}{56}=\dfrac{3}{7}$

$\triangle P_nP_{n+1}P_{n+2}$ $(n\geq 1)$により増加する面積は

$\dfrac{3}{7}\left\{\left(\dfrac{1}{2}\right)^n\dfrac{\sqrt{3}}{2}\left(\dfrac{1}{2}\right)^{n+1}\dfrac{1}{2}+\left(\dfrac{1}{2}\right)^{n+1}\dfrac{\sqrt{3}}{2}\left(\dfrac{1}{2}\right)^{n+2}\dfrac{1}{2}-\left(\dfrac{1}{2}\right)^n\dfrac{\sqrt{3}}{2}\left(\dfrac{1}{2}\right)^{n+2}\dfrac{1}{2}\right\}=\dfrac{3}{7}\dfrac{\sqrt{3}}{8}\left(\dfrac{1}{4}\right)^n\left(1+\dfrac{1}{4}-\dfrac{1}{2}\right)=\dfrac{9\sqrt{3}}{224}\left(\dfrac{1}{4}\right)^n$

$\triangle P_0P_1P_2$の面積は(1)より $\dfrac{1}{2}\dfrac{9\sqrt{3}+3\sqrt{3}}{64}=\dfrac{3\sqrt{3}}{32}$

$S_n=\dfrac{3\sqrt{3}}{32}+\dfrac{9\sqrt{3}}{224}\left\{\left(\dfrac{1}{4}\right)+\left(\dfrac{1}{4}\right)^2+\cdots+\left(\dfrac{1}{4}\right)^n\right\}=\dfrac{3\sqrt{3}}{32}+\dfrac{9\sqrt{3}}{224}\dfrac{\frac{1}{4}\left\{1-\left(\frac{1}{4}\right)^n\right\}}{1-\frac{1}{4}}$

$\displaystyle\lim_{n\to\infty}S_n=\dfrac{3\sqrt{3}}{32}+\dfrac{3\sqrt{3}}{224}=\dfrac{21\sqrt{3}+3\sqrt{3}}{224}=\dfrac{3\sqrt{3}}{28}$