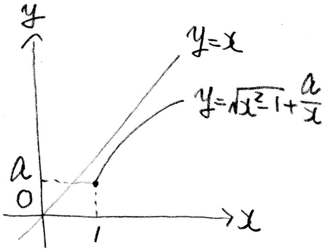
東大理科　1992後期　①

(1)



$x \geqq 1$ のとき $x > \sqrt{x^2-1}$ より, $a \leqq 0$ のとき, $y<x$ に含まれる.

$x=1$ のとき $\sqrt{x^2-1}+\frac{a}{x}=a$ より $a \geqq 1$ のとき, $y<x$ に含まれない.

よって $0<a<1$ のときを調べる.

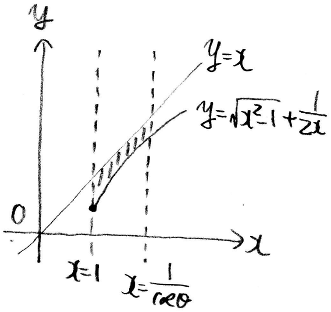
$\sqrt{x^2-1}+\frac{a}{x}<x$ が成り立てばよい.

$\sqrt{x^2-1}<x-\frac{a}{x}$, $x-\frac{a}{x}>0$ より $\cancel{x^2}-1<\cancel{x^2}-2a+\frac{a^2}{x^2}$　$\frac{2a-1}{a^2}<\frac{1}{x^2}$

これが任意のxについて成り立つには $a \leqq \frac{1}{2}$ であればよい.

よって $a_0=\frac{1}{2}$

(2)



$$V(\theta)=\int_1^{\frac{1}{\cos\theta}}\pi\left\{x^2-\left(\sqrt{x^2-1}+\frac{1}{2x}\right)^2\right\}dx=\pi\int_1^{\frac{1}{\cos\theta}}\left(\cancel{x^2}-\cancel{x^2}+1-\frac{\sqrt{x^2-1}}{x}-\frac{1}{4x^2}\right)dx$$

$$=\pi\left[x+\frac{1}{4x}\right]_1^{\frac{1}{\cos\theta}}-\pi\int_1^{\frac{1}{\cos\theta}}\sqrt{1-\frac{1}{x^2}}\,dx=\pi\left(\frac{1}{\cos\theta}+\frac{\cos\theta}{4}-\frac{5}{4}\right)-\pi\int_0^{\theta}\sqrt{1-\cos^2\varphi}\,\frac{\sin\varphi}{\cos^2\varphi}d\varphi$$

$\frac{1}{x}=\cos\varphi$ とおく. $\begin{array}{c|c} x & 1\to\frac{1}{\cos\theta} \\ \varphi & 0\to\theta \end{array}$, $\frac{dx}{d\varphi}=\frac{\sin\varphi}{\cos^2\varphi}$

$$=\pi\left(\frac{1}{\cos\theta}+\frac{\cos\theta}{4}-\frac{5}{4}\right)-\pi\int_0^{\theta}\frac{1-\cos^2\varphi}{\cos^2\varphi}d\varphi=\pi\left(\frac{1}{\cos\theta}+\frac{\cos\theta}{4}-\frac{5}{4}\right)-\pi\left[\tan\varphi-\varphi\right]_0^{\theta}$$

※ $\frac{d}{d\varphi}\tan\varphi=\frac{\cos\varphi\cos\varphi+\sin\varphi\sin\varphi}{\cos^2\varphi}=\frac{1}{\cos^2\varphi}$

$$=\pi\left(\frac{1}{\cos\theta}+\frac{\cos\theta}{4}-\frac{5}{4}\right)-\pi(\tan\theta-\theta)=\pi\left(\frac{1-\sin\theta}{\cos\theta}+\frac{\cos\theta}{4}+\theta-\frac{5}{4}\right)$$

(3) $\frac{1-\sin\theta}{\cos\theta}=\frac{(1-\sin\theta)(1+\sin\theta)}{\cos\theta(1+\sin\theta)}=\frac{\cos\theta}{1+\sin\theta}$ より $\lim_{\theta\to\frac{\pi}{2}-0}V(\theta)=\pi\left(\frac{\pi}{2}-\frac{5}{4}\right)$