東大理科　1993前期　④

$$f(x)^2=(x^n+px+q)^2=x^{2n}+p^2x^2+q^2+2px^{n+1}+2pqx+2qx^n$$

(i) nが偶数のとき

$$\int_{-1}^{1}f(x)^2dx=2\int_0^1(x^{2n}+p^2x^2+q^2+2qx^n)dx=2\left[\frac{x^{2n+1}}{2n+1}+p^2\frac{x^3}{3}+q^2x+2q\frac{x^{n+1}}{n+1}\right]_0^1=2\left(\frac{1}{2n+1}+\frac{p^2}{3}+q^2+\frac{2q}{n+1}\right)$$

$$I=\frac{p^2}{3}+q^2+\frac{2q}{n+1}+\frac{1}{(n+1)^2}-\frac{1}{(n+1)^2}+\frac{1}{2n+1}=\frac{p^2}{3}+\left(q+\frac{1}{n+1}\right)^2+\frac{n^2+\cancel{2n}+\cancel{1}-\cancel{2n}-\cancel{1}}{(2n+1)(n+1)^2}=\frac{p^2}{3}+\left(q+\frac{1}{n+1}\right)^2+\frac{n^2}{(2n+1)(n+1)^2}$$

よって $(p,q)=\left(0,-\frac{1}{n+1}\right)$ のとき Iは最小値 $\frac{n^2}{(2n+1)(n+1)^2}$ をとる.

(ii) nが奇数のとき

$$\int_{-1}^{1}f(x)^2dx=2\int_0^1(x^{2n}+p^2x^2+q^2+2px^{n+1})dx=2\left[\frac{x^{2n+1}}{2n+1}+p^2\frac{x^3}{3}+q^2x+2p\frac{x^{n+2}}{n+2}\right]_0^1=2\left(\frac{1}{2n+1}+\frac{p^2}{3}+q^2+\frac{2p}{n+2}\right)$$

$$I=\frac{1}{3}\left\{p^2+\frac{6p}{n+2}+\frac{9}{(n+2)^2}\right\}-\frac{3}{(n+2)^2}+q^2+\frac{1}{2n+1}=\frac{1}{3}\left(p+\frac{3}{n+2}\right)^2+q^2+\frac{n^2+4n+4-6n-3}{(2n+1)(n+2)^2}=\frac{1}{3}\left(p+\frac{3}{n+2}\right)^2+q^2+\frac{(n-1)^2}{(2n+1)(n+2)^2}$$

よって $(p,q)=\left(-\frac{3}{n+2},0\right)$ のとき Iは最小値 $\frac{(n-1)^2}{(2n+1)(n+2)^2}$ をとる.