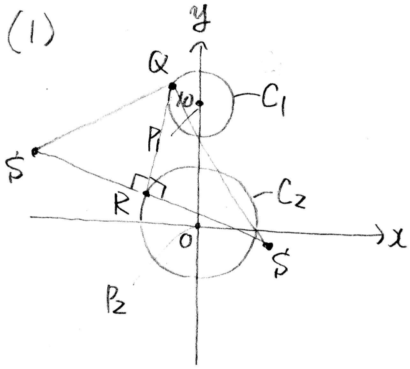
東大理科 1998後期 ①

(1)



Q, Rの座標は $(\cos\theta, 10+\sin\theta)$, $(2\cos\varphi, 2\sin\varphi)$ とおける

$\overrightarrow{RQ}=(\cos\theta-2\cos\varphi, \sin\theta-2\sin\varphi+10)$

Sの座標は

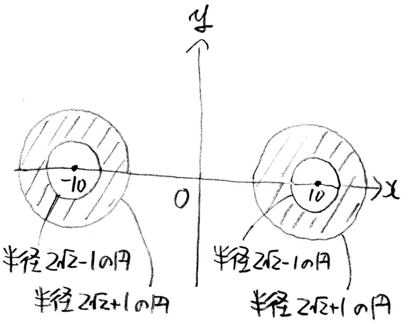
$$\begin{pmatrix}2\cos\varphi\\2\sin\varphi\end{pmatrix}+\begin{pmatrix}\cos\frac{\pi}{2} & -\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{pmatrix}\begin{pmatrix}\cos\theta-2\cos\varphi\\ \sin\theta-2\sin\varphi+10\end{pmatrix}=\begin{pmatrix}2\cos\varphi\\2\sin\varphi\end{pmatrix}+\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\cos\theta-2\cos\varphi\\ \sin\theta-2\sin\varphi+10\end{pmatrix}$$

$$=\begin{pmatrix}-\sin\theta+2\sin\varphi+2\cos\varphi-10\\ \cos\theta+2\sin\varphi-2\cos\varphi\end{pmatrix}=\begin{pmatrix}\cos(\theta+\frac{\pi}{2})\\ \sin(\theta+\frac{\pi}{2})\end{pmatrix}+\begin{pmatrix}\cos(\varphi-\frac{\pi}{2}) & -\sin(\varphi-\frac{\pi}{2})\\ \sin(\varphi-\frac{\pi}{2}) & \cos(\varphi-\frac{\pi}{2})\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}+\begin{pmatrix}-10\\0\end{pmatrix} \quad -①$$

または

$$\begin{pmatrix}2\cos\varphi\\2\sin\varphi\end{pmatrix}+\begin{pmatrix}\cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2})\\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2})\end{pmatrix}\begin{pmatrix}\cos\theta-2\cos\varphi\\ \sin\theta-2\sin\varphi+10\end{pmatrix}=\begin{pmatrix}2\cos\varphi\\2\sin\varphi\end{pmatrix}+\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\cos\theta-2\cos\varphi\\ \sin\theta-2\sin\varphi+10\end{pmatrix}$$

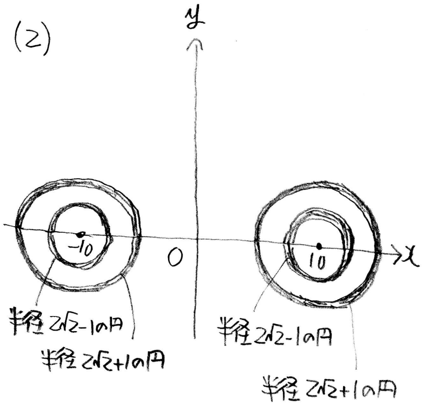
$$=\begin{pmatrix}\sin\theta-2\sin\varphi+2\cos\varphi+10\\ -\cos\theta+2\sin\varphi+2\cos\varphi\end{pmatrix}=\begin{pmatrix}\cos(\theta-\frac{\pi}{2})\\ \sin(\theta-\frac{\pi}{2})\end{pmatrix}+\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}+\begin{pmatrix}10\\0\end{pmatrix} \quad -②$$



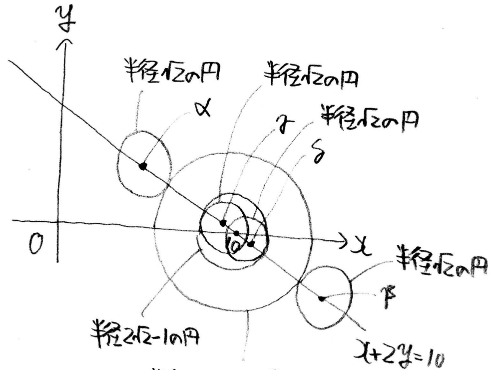
①または②で表される点は左図の斜線部上にある.

よって、Sの軌跡は左図の斜線部, 境界線上の点含む.

(2)



Q, Rがただ一つだけ定まるSの集合は左図の太線部



よって P_3 は左図の α, β, γ, δ

α, βは $x+2y=10$ と $(x-10)^2+y^2=(2\sqrt{2}+1+\sqrt{2})^2$ の交点であるから,

$5y^2=(3\sqrt{2}+1)^2$, $y=\pm\frac{3\sqrt{2}+1}{\sqrt{5}}=\pm\frac{3\sqrt{10}+\sqrt{5}}{5}$, $x=10\mp\frac{6\sqrt{10}+2\sqrt{5}}{5}$ より,

$\left(10-\frac{6\sqrt{10}+2\sqrt{5}}{5}, \frac{3\sqrt{10}+\sqrt{5}}{5}\right)$, $\left(10+\frac{6\sqrt{10}+2\sqrt{5}}{5}, -\frac{3\sqrt{10}+\sqrt{5}}{5}\right)$

γ, δは $x+2y=10$ と $(x-10)^2+y^2=(2\sqrt{2}-1-\sqrt{2})^2$ の交点であるから

$5y^2=(\sqrt{2}-1)^2$, $y=\pm\frac{\sqrt{2}-1}{\sqrt{5}}=\pm\frac{\sqrt{10}-\sqrt{5}}{5}$, $x=10\mp\frac{2\sqrt{10}-2\sqrt{5}}{5}$ より

$\left(10-\frac{2\sqrt{10}-2\sqrt{5}}{5}, \frac{\sqrt{10}-\sqrt{5}}{5}\right)$, $\left(10+\frac{2\sqrt{10}-2\sqrt{5}}{5}, -\frac{\sqrt{10}-\sqrt{5}}{5}\right)$

よって、P_3 の座標は $\left(10-\frac{6\sqrt{10}+2\sqrt{5}}{5}, \frac{3\sqrt{10}+\sqrt{5}}{5}\right)$, $\left(10+\frac{6\sqrt{10}+2\sqrt{5}}{5}, -\frac{3\sqrt{10}+\sqrt{5}}{5}\right)$, $\left(10-\frac{2\sqrt{10}-2\sqrt{5}}{5}, \frac{\sqrt{10}-\sqrt{5}}{5}\right)$, $\left(10+\frac{2\sqrt{10}-2\sqrt{5}}{5}, -\frac{\sqrt{10}-\sqrt{5}}{5}\right)$