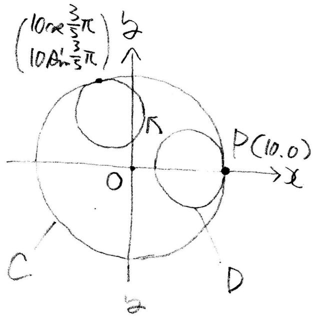
東大理科　2004年前期　③

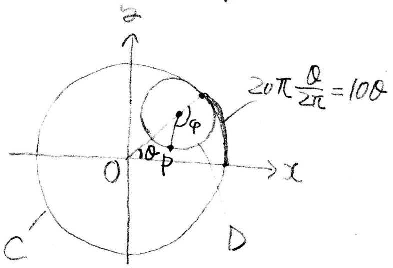


xy平面上に左図のようにC, Dをとる

初め、Pは(10.0)にあるとする

$\frac{2\pi}{20\pi}=\frac{\theta}{6\pi}$ (角)(円周)、$\theta=\frac{3}{5}\pi$ より

Pが再びCに接するとき Pは $(10\cos\frac{3}{5}\pi, 10\sin\frac{3}{5}\pi)$ にある



左図のような状態のとき $\frac{2\pi}{6\pi}=\frac{\varphi}{10\theta}$ (角)(円周)、$\varphi=\frac{10}{3}\theta$ より

Pの座標は $(7\cos\theta+3\cos(-\frac{7}{3}\theta), 7\sin\theta+3\sin(-\frac{7}{3}\theta))$

$(7\cos\theta+3\cos\frac{7}{3}\theta, 7\sin\theta-3\sin\frac{7}{3}\theta)$, $0<\theta<\frac{3}{5}\pi$ である

$x=7\cos\theta+3\cos\frac{7}{3}\theta$

$x'=-7\sin\theta-3\sin\frac{7}{3}\theta\cdot\frac{7}{3}=-7(\sin\theta+\sin\frac{7}{3}\theta)=-14\sin\frac{\frac{10}{3}\theta}{2}\cos\frac{-\frac{4}{3}\theta}{2}=-14\sin\frac{5}{3}\theta\cos\frac{2}{3}\theta<0$

※ $\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

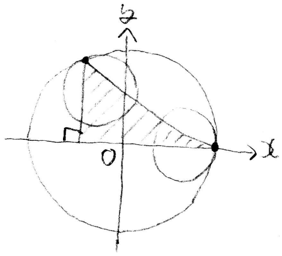
よってPのx座標は単調減少

$y=7\sin\theta-3\sin\frac{7}{3}\theta$

$y'=7\cos\theta-3\cos\frac{7}{3}\theta\cdot\frac{7}{3}=7(\cos\theta-\cos\frac{7}{3}\theta)=-14\sin\frac{\frac{10}{3}\theta}{2}\sin\frac{-\frac{4}{3}\theta}{2}=14\sin\frac{5}{3}\theta\sin\frac{2}{3}\theta>0$

※ $\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

よってPのy座標は単調増加



上図の斜線部の面積をSとすると

$$S=\int_{10\cos\frac{3}{5}\pi}^{10}y\,dx=\int_{\frac{3}{5}\pi}^{0}y(\theta)\frac{dx(\theta)}{d\theta}d\theta=\int_{\frac{3}{5}\pi}^{0}(7\sin\theta-3\sin\tfrac{7}{3}\theta)(-7\sin\theta-3\sin\tfrac{7}{3}\theta\cdot\tfrac{7}{3})d\theta=7\int_{0}^{\frac{3}{5}\pi}(7\sin\theta-3\sin\tfrac{7}{3}\theta)(\sin\theta+\sin\tfrac{7}{3}\theta)d\theta$$

$$=7\int_0^{\frac{3}{5}\pi}(7\sin^2\theta+4\sin\theta\sin\tfrac{7}{3}\theta-3\sin^2\tfrac{7}{3}\theta)d\theta=7\int_0^{\frac{3}{5}\pi}\left(7\frac{1-\cos2\theta}{2}-2\cos\tfrac{10}{3}\theta+2\cos\tfrac{4}{3}\theta-3\frac{1-\cos\frac{14}{3}\theta}{2}\right)d\theta$$

※ $\sin A\sin B=-\frac{1}{2}\{\cos(A+B)-\cos(A-B)\}$

$$=7\int_0^{\frac{3}{5}\pi}\left(\tfrac{7}{2}-\tfrac{7}{2}\cos2\theta-2\cos\tfrac{10}{3}\theta+2\cos\tfrac{4}{3}\theta-\tfrac{3}{2}+\tfrac{3}{2}\cos\tfrac{14}{3}\theta\right)d\theta=7\left[2\theta-\tfrac{7}{2}\tfrac{1}{2}\sin2\theta-2\tfrac{3}{10}\sin\tfrac{10}{3}\theta+2\tfrac{3}{4}\sin\tfrac{4}{3}\theta+\tfrac{3}{2}\tfrac{3}{14}\sin\tfrac{14}{3}\theta\right]_0^{\frac{3}{5}\pi}$$

$$=7\left(\tfrac{6}{5}\pi-\tfrac{7}{4}\sin\tfrac{6}{5}\pi-\tfrac{3}{5}\sin2\pi+\tfrac{3}{2}\sin\tfrac{4}{5}\pi+\tfrac{9}{28}\sin\tfrac{14}{5}\pi\right)=7\left(\tfrac{6}{5}\pi+\tfrac{7}{4}\sin\tfrac{1}{5}\pi+\tfrac{3}{2}\sin\tfrac{1}{5}\pi+\tfrac{9}{28}\sin\tfrac{1}{5}\pi\right)=\tfrac{42}{5}\pi+7\left(\tfrac{49+42+9}{28}\right)\sin\tfrac{1}{5}\pi=\tfrac{42}{5}\pi+25\sin\tfrac{1}{5}\pi$$

大きいほうの面積は $100\pi\frac{\frac{7}{5}\pi}{2\pi}+S-\frac{1}{2}(-10\cos\frac{3}{5}\pi)10\sin\frac{3}{5}\pi=70\pi+\frac{42}{5}\pi+\cancel{25\sin\frac{1}{5}\pi}+\cancel{25\sin\frac{6}{5}\pi}=\frac{392}{5}\pi$

小さいほうの面積は $100\pi-\frac{392}{5}\pi=\frac{108}{5}\pi$