東大理科 2007前期 ④

(1) $PA = aP^2 + (a+1)PQ = aP$

$QA = aQA + (a+1)Q^2 = (a+1)Q$

よって $(P+Q)A = PA + QA = aP + (a+1)Q = A$

(2) (1)の5つの条件をすべてみたす行列 P, Q が存在するとする

$P+Q = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ とする

$(P+Q)A = A$ より $\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} a & 0 \\ 1 & a+1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 1 & a+1 \end{pmatrix}$, $\begin{pmatrix} ax+y & (a+1)y \\ az+w & (a+1)w \end{pmatrix} = \begin{pmatrix} a & 0 \\ 1 & a+1 \end{pmatrix}$ $a \neq -1, a \neq 0$ より $\begin{cases} x=1 \\ y=0 \\ z=0 \\ w=1 \end{cases}$

$P = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}$ $Q = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}$ とする. $P+Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ より $\begin{cases} q_1 = -p_1+1 \\ q_2 = -p_2 \\ q_3 = -p_3 \\ q_4 = -p_4+1 \end{cases}$

$A = aP + (a+1)Q$ より $\begin{pmatrix} a & 0 \\ 1 & a+1 \end{pmatrix} = \begin{pmatrix} ap_1 & ap_2 \\ ap_3 & ap_4 \end{pmatrix} + \begin{pmatrix} (a+1)(-p_1+1) & -(a+1)p_2 \\ -(a+1)p_3 & (a+1)(-p_4+1) \end{pmatrix}$

$\begin{pmatrix} a & 0 \\ 1 & a+1 \end{pmatrix} = \begin{pmatrix} -p_1+a+1 & -p_2 \\ -p_3 & -p_4+a+1 \end{pmatrix}$ $\begin{cases} p_1 = 1 \\ p_2 = 0 \\ p_3 = -1 \\ p_4 = 0 \end{cases}$ $\begin{cases} q_1 = 0 \\ q_2 = 0 \\ q_3 = 1 \\ q_4 = 1 \end{cases}$

逆に $P = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ のとき $aP + (a+1)Q = \begin{pmatrix} a & 0 \\ -a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a+1 & a+1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 1 & a+1 \end{pmatrix} = A$

$P^2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = P$, $Q^2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = Q$

$PQ = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$, $QP = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$

よって $P = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

(3) $P = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ とすると $A_k = kP + (k+1)Q$, $P^2 = P$, $Q^2 = Q$, $PQ = O$, $QP = O$

$A_n A_{n-1} A_{n-2} \cdots A_2 = \{nP + (n+1)Q\}\{(n-1)P + nQ\}\{(n-2)P + (n-1)Q\} \cdots (2P+3Q)$

$= n!P + \frac{(n+1)!}{2}Q = \begin{pmatrix} n! & 0 \\ -n! & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{(n+1)!}{2} & \frac{(n+1)!}{2} \end{pmatrix} = \begin{pmatrix} n! & 0 \\ \frac{n-1}{2}n! & \frac{n+1}{2}n! \end{pmatrix}$