東大文科　2008前期 ①

$\int_{-1}^{1} f(x)dx = 1$ より $\left[\frac{x^3}{3} - (\alpha+\beta)\frac{x^2}{2} + \alpha\beta x\right]_{-1}^{1} = 1$，$\frac{1}{3} - \cancel{(\alpha+\beta)\frac{1}{2}} + \alpha\beta + \frac{1}{3} + \cancel{(\alpha+\beta)\frac{1}{2}} + \alpha\beta = 1$，$2\alpha\beta = \frac{1}{3}$．$\alpha\beta = \frac{1}{6}$

$S = \left[\frac{x^3}{3} - (\alpha+\beta)\frac{x^2}{2} + \frac{1}{6}x\right]_0^{\alpha} = \frac{1}{3}\alpha^3 - \frac{1}{2}\alpha^3 - \frac{1}{2}\alpha\frac{1}{6} + \frac{1}{6}\alpha = -\frac{1}{6}\alpha^3 + \frac{1}{12}\alpha$

$S' = -\frac{1}{2}\alpha^2 + \frac{1}{12}$．$S' = 0$ のとき $6\alpha^2 = 1$　$\alpha \geqq 0$ より $\alpha = \frac{1}{\sqrt{6}}$　このとき $\beta = \frac{1}{\sqrt{6}}$ であるから $\alpha \leqq \beta$ は成り立つ

α	0	…	$\frac{1}{\sqrt{6}}$	…
S'		+	0	−
S		↗	$\frac{\sqrt{6}}{108}$	↘

S の増減表は左表

よって，求める値は $\frac{\sqrt{6}}{108}$

※ $S\left(\frac{1}{\sqrt{6}}\right) = -\frac{1}{6}\frac{1}{6\sqrt{6}} + \frac{1}{12}\frac{1}{\sqrt{6}}$

$= \frac{-1+3}{36\sqrt{6}} = \frac{1}{18\sqrt{6}} = \frac{\sqrt{6}}{108}$